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**LETTER TO THE EDITOR**

# Long-range percolation in one dimension

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**Abstract.** The problem of long-range percolation in one dimension is proposed. We consider a one-dimensional bond percolation system with bonds connecting an infinite number of neighbours where the occupation probability for the $n$th nearest-neighbour bond $p_n$ varies as $p_1/n^s$. Using the transfer-matrix method, we find that when $s>2$ only the short-range percolation exists; namely, the system percolates only when $p_1=1$. A transition to long-range percolation is found at $s=2$ where the percolation threshold drops suddenly from the short-range value $p_1^c=1$ to the long-range value $p_1^c=0$.

In the past decade, the percolation problem (for reviews see Stauffer 1979 and Essam 1980) has aroused considerable interest because of its close relationship with the thermal critical phenomena (Kasteleyn and Fortuin 1969). Percolation in one-dimensional systems is one of the few cases where exact solutions can be obtained. Recently both site and bond percolation in one dimension with bonds connecting $L$th nearest neighbours have been solved by various methods, including the generating function (Klein *et al* 1978), the renormalisation group (Reynolds *et al* 1980 and Li *et al* 1983) and the transfer-matrix methods (Zhang and Shen 1982, Zhang *et al* 1983). Although the critical percolation probability is trivial in one dimension, the critical exponents are found to be $L$-dependent. Such a 'bond range' dependence of the critical behaviour is related to the corresponding 'thermal' problem with multi-spin interaction (Klein *et al* 1978).

In this letter, we propose a problem of long-range percolation in one dimension. We consider a one-dimensional bond percolation system with bonds connecting an infinite number of neighbours. If the occupation probability for the $n$th nearest-neighbour bond $p_n$ varies as $p_1/n^s$, then the following question can be asked. For any given value of $p_1$ in the range $0<p_1<1$, is there a critical value $s_c(p_1)$ of $s$ such that the system can percolate without requiring $p_1=1$?. This problem is analogous to the famous one-dimensional Ising ferromagnet with an interaction energy which varies as $J(n)=n^{-\alpha}$. For this model, it is well known that the system is completely ordered at all temperatures for $0\leqslant\alpha\leqslant1$, and disordered at all temperatures for $\alpha>2$ (Ruelle 1968). Dyson (1969) has proved that a phase transition exists for $1<\alpha<2$. The existence of a transition for $\alpha=2$ has been conjectured by Thouless (1969) and was proved very recently by Fröhlich and Spencer (1982).

Before discussing the long-range percolation, we first consider a one-dimensional bond percolation system with bonds connecting $L$th nearest neighbours. The critical behaviours of such a system can be found by using the transfer-matrix method (Zhang and Shen 1982, Zhang *et al* 1983). Here we only give a brief account of the method; further details may be found in the original papers.

We divide the chain into overlapping columns each containing $L$ sites. If we take the sites $1, 2, \ldots, L-1$ and $L$ as the $N$th column, then the $(N+1)$th column contains sites $2, 3, \ldots, L$ and $(L+1)$ (for $L=3$, see figure 1). To each $i$th site of the $N$th column, we assign a value $m_i$ which has the value 1 or 0 depending on whether the $i$th site is connected to or disconnected from the first column. If $P_{m_1 m_2 \ldots m_L}(N)$ is the probability of being in the configuration $(m_1, m_2, \ldots, m_L)$, then the transfer matrix $T^{(L)}$, with dimensionality $2^L \times 2^L$, is defined by

$$P_{n_1 n_2 \ldots n_L}(N+1) = \sum_{m_1, \ldots m_L = 0,1} \langle n_1 n_2 \ldots n_L | T^{(L)} | m_1 m_2 \ldots m_L \rangle P_{m_1 m_2 \ldots, m_L}(N). \tag{1}$$

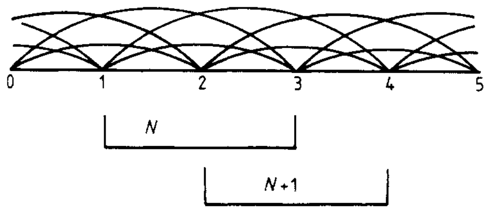


**Figure 1.** A linear chain with bonds connecting third nearest neighbours ($L=3$). Sites (1, 2, 3) and (2, 3, 4) are taken to be the $N$th and $(N+1)$th columns respectively.

The correlation length $\xi$ is related to the largest nontrivial eigenvalue $\lambda_m^{(L)}$ of the transfer matrix by the relation (Derrida and Vannimenus 1980)

$$\xi = -1/\ln \lambda_m^{(L)}. \tag{2}$$

The critical percolation is given by the condition $\lambda_m^{(L)} = 1$ where the correlation length becomes infinite. Writing $p_i$ as the occupation probability for the $i$th nearest-neighbour bond and $q_i = 1 - p_i$, the transfer matrix $T^{(L)}$ in (1) has the following form

$$\begin{aligned}
&\langle n_1 n_2 \ldots n_L | T^{(L)} | m_1 m_2 \ldots m_L \rangle \\
&\quad = \delta_{n_1 m_2} \ldots \delta_{n_{L-1} m_L} [q_L^{m_1} \ldots q_1^{m_L} \delta_{n_L 0} + (1 - q_L^{m_1} \ldots q_1^{m_L}) \delta_{n_L 1}] \\
&\qquad\qquad\qquad\qquad \text{if } (m_1, m_2, \ldots, m_L) \neq (0, 0, \ldots, 0) \\
&\quad = \delta_{n_1 m_2} \ldots \delta_{n_{L-1} m_L} \delta_{n_L 0} \qquad \text{if } (m_1, m_2, \ldots, m_L) = (0, 0, \ldots, 0).
\end{aligned} \tag{3}$$

Using the proper labelling procedure, (3) can be cast into a systematic duo-diagonal form (Domb 1949). When $L=3$, the transfer matrix $T^{(3)}$ has the form

$$T^{(3)} = \begin{bmatrix}
1 & 0 & 0 & 0 & q_3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1-q_3 & 0 & 0 & 0 \\
0 & q_1 & 0 & 0 & 0 & q_1 q_3 & 0 & 0 \\
0 & 1-q_1 & 0 & 0 & 0 & 1-q_1 q_3 & 0 & 0 \\
0 & 0 & q_2 & 0 & 0 & 0 & q_2 q_3 & 0 \\
0 & 0 & 1-q_2 & 0 & 0 & 0 & 1-q_2 q_3 & 0 \\
0 & 0 & 0 & q_1 q_2 & 0 & 0 & 0 & q_1 q_2 q_3 \\
0 & 0 & 0 & 1-q_1 q_2 & 0 & 0 & 0 & 1-q_1 q_2 q_3
\end{bmatrix}. \tag{4}$$

For any $L$, $T^{(L)}$ has the form

$$T^{(L)} = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \times 0 & \dots & 0 & 0 \\ 0 & & & & & & & \\ 0 & & & & & & & \\ \vdots & & & & R^{(L)} & & & \\ 0 & & & & & & & \\ 0 & & & & & & & \end{bmatrix} \tag{5}$$

where $R^{(L)}$ is a matrix of dimensionality $(2^L-1)\times(2^L-1)$. The largest nontrivial eigenvalue $\lambda_m^{(L)}$ of $T^{(L)}$ is thus the largest root of the characteristic equation of $R^{(L)}$: $\det(R^{(L)}-\lambda I)\equiv A^{(L)}(\lambda;q_1,q_2,\dots,q_L)=0$. In the most general case, the matrix $R^{(L)}$ is irreducible. From the Perron–Frobenius theorem, we know that the largest root of a non-negative irreducible matrix is non-degenerate (Rosenblatt 1962). We can write

$$A^{(L)}(\lambda;q_1,q_2,\dots,q_L)\equiv A^{(L)}(\lambda;\boldsymbol{q})=(\lambda_m^{(L)}-\lambda)B^{(L)}(\lambda;\boldsymbol{q}). \tag{6}$$

Putting $\lambda=1$ in (6), we have

$$\lambda_m^{(L)}=1+[A^{(L)}(1;\boldsymbol{q})/B^{(L)}(1;\boldsymbol{q})]. \tag{7}$$

It is extremely difficult to find the explicit expressions of the functions $A^{(L)}(1;\boldsymbol{q})$ and $B^{(L)}(1;\boldsymbol{q})$ for general $L$. However, some properties of these functions are known. We have proved rigorously that, for any given $L$, $A^{(L)}(1;\boldsymbol{q})$ has the following form

$$A^{(L)}(1,\boldsymbol{q})=-q_1q_2^2\dots q_L^L(1+q_1f_L(\boldsymbol{q})). \tag{8}$$

The proof of (8) which is rather long and complicated will be given elsewhere. Since $T^{(L)}$ is a stochastic matrix, all the eigenvalues $\lambda_i^{(L)}$ have the property $|\lambda_i^{(L)}|\leq 1$ (Pearl 1973). Excluding the trivial $\lambda=1$ and $\lambda_m^{(L)}$, $B^{(L)}(\lambda;\boldsymbol{q})$ contains all the other $2^L-2$ roots. Except for some special cases where the system can be decoupled into many independent chains, it can be shown that both the functions $B^{(L)}(1;\boldsymbol{q})$ and $(1+q_1f_L(\boldsymbol{q}))$ are positive. From (2), (7) and (8), expanding $\ln\lambda_m^{(L)}$ to the lowest order in $q$, we have

$$\xi\approx(q_1q_2^2\dots q_L^L)^{-1}[B^{(L)}(1;\boldsymbol{q})/(1+q_1f_L(\boldsymbol{q}))]. \tag{9}$$

The critical surface is then given by

$$g(\boldsymbol{q})\equiv q_1q_2^2\dots q_L^L=0. \tag{10}$$

Various correlation length exponents $\nu$ can be obtained by approaching the critical surface in different ways. For instance, if we let all the $q$'s be equal then we have $\nu=(1+2+3+\dots+L)=L(L+1)/2$. This result has been obtained previously by Zhang *et al* (1983) and Li *et al* (1983). When all the $q$'s are independent, very many critical phenomena can be found. This has been studied in detail for $L\leq 3$ (Zhang and Shen 1982) and can now be generalised to the case of general $L$.

When the system can be decoupled into many independent chains, the functions $B^{(L)}(1;\boldsymbol{q})$ and $(1+q_1f_L(\boldsymbol{q}))$ might become zero. For instance, when only one of the $q$'s, say $q_i$ ($i=2,3,\dots,L$), is not equal to 1 while all the other $q$'s are equal to 1, the system is decoupled into $i$ independent chains with nearest-neighbour bonds only. In these cases, both the functions $B^{(L)}(1;\boldsymbol{q})$ and $(1+q_1f_L(\boldsymbol{q}))$ might contain factors of $q_i$ to certain powers which cancel the $q_i^i$ factor in front and give the correct exponent $\nu=1$. Except for those cases described above, $\nu$ is determined solely by the factor $g(\boldsymbol{q})$.

To study the possibility of long-range percolation, we first assume that the occupation probability of the $n$th nearest-neighbour bond $p_n$ varies as $p_1/n^s$ and then let $L$ go to infinity. Putting $q_n = 1-p_1/n^s$ in (7), $\lambda_m^{(L)}$ can be written as

$$\lambda_m^{(L)} = 1+\frac{\tilde{A}^{(L)}(p_1, s)}{\tilde{B}^{(L)}(p_1, s)} = 1-\frac{(1-p_1)\Pi_{n=2}^{L}(1-p_1/n^s)^n[1+(1-p_1)\tilde{f}_L(p_1, s)]}{\tilde{B}^{(L)}(p_1, s)}. \tag{11}$$

For any given $L$, this system can percolate only when $p_1=1$ where $\lambda_m^{(L)}$ is equal to 1. As $L$ goes to infinity, we define the functions $\tilde{f}(p_1, s)$ and $\tilde{B}(p_1, s)$ as

$$\begin{aligned} \tilde{f}(p_1, s) &\equiv \lim_{L\to\infty} \tilde{f}_L(p_1, s) \\ \tilde{B}(p_1, s) &\equiv \lim_{L\to\infty} \tilde{B}^{(L)}(p_1, s) \end{aligned} \tag{12}$$

and $\lambda_m$ can be written as

$$\lambda_m \equiv \lim_{L\to\infty} \lambda_m^{(L)} = 1-C(p_1, s)D(p_1, s) \tag{13}$$

where

$$\begin{aligned} C(p_1, s) &= (1-p_1)Z(p_1, s) \\ Z(p_1, s) &= \prod_{n=2}^{\infty} (1-p_1/n^s)^n \\ D(p_1, s) &= [1+(1-p_1)\tilde{f}(p_1, s)]/\tilde{B}(p_1, s). \end{aligned} \tag{14}$$

The existence of $\lambda_m$ in (13) can be seen from the following facts. As long as $s$ is positive, $p_L$ approaches zero when $L$ goes to infinity. It can be proved that when $p_L=0$ the characteristic equation of $T^{(L)}$ is simply the characteristic equation of $T^{(L-1)}$ with additional zero root of order $(2L-2^{L-1})$ (i.e. $\lambda^{2L-1}=0$). It is also known that when $L$ is increased, additional bonds which are added to the system will increase the correlation length for any fixed values of $p_1$ and $s$. So, $\lambda_m^{(L)}$ is a monotonically increasing function of $L$ and will approach a limiting value as $L$ goes to infinity.

Since $Z(p_1, s)$ is known explicitly, we can analyse its analytic properties. Taking the logarithm of $Z(p_1, s)$, we have

$$-\ln Z(p_1, s) = -\sum_{n=2}^{\infty} n \ln\left(1-\frac{p_1}{n^s}\right)^n = \sum_{n=2}^{\infty}\sum_{m=1}^{\infty} \frac{p_1^m}{mn^{ms-1}}. \tag{15}$$

For any value of $p_1$ in the range $0<p_1<1$, it can be proved that the double sum in (15) diverges when $s \leqslant 2$ and converges when $s>2$. When $s>2$, changing the order of summation in (15), we find

$$\begin{aligned} -\ln Z(p_1, s) &= \sum_{m=1}^{\infty}\left\{\sum_{n=1}^{\infty}\frac{p_1^m}{mn^{ms-1}}-\frac{p_1^m}{m}\right\} \\ &= \sum_{m=1}^{\infty}\frac{p_1^m}{m}\zeta(ms-1)+\ln(1-p_1) \end{aligned} \tag{16}$$

where $\zeta(x)$ is the Riemann zeta function. Since the function $\zeta(x)$ has a simple pole at $x=1$ with residue 1 in the entire complex plane $x$ (Abramowitz and Stegun 1970), we find, from (16), that the singularity of $-\ln Z(p_1, s)$ at $s=2$ is also a simple pole with residue $p_1$. Equivalently, the singularity of $Z(p_1, s)$ at $s=2$ is an essential singularity of the form

$$Z_{\text{sing}}(p_1, s) \sim \exp[-p_1/(s-2)].$$

So, when $s$ is greater than 2, we know, from (13) and (14), that the system can only percolate with $p_1 = 1$ where $\lambda_m = 1$. As the value of $s$ decreases to 2, because $Z(p_1, 2) = 0$ and $\lambda_m = 1$, the system percolates for all values of $p_1$ in the range $0 < p_1 < 1$. We call this a long-range percolation where the system percolates through an infinite number of long-range bonds without requiring that $p_1 = 1$. Since the function $D(p_1, s)$ is not known explicitly, we have assumed that $D(p_1, s)$ is analytic for all values of $s$ in the region $s \geq 2$.

Comparing our results with the one-dimensional Ising ferromagnet with $J(n) = n^{-s}$ interaction energy, it is interesting to see that the same critical value $s = 2$ is found in the percolation system where long-range percolation is realised. At this critical value, the percolation threshold $p_1^c$ drops suddenly from the short-range value $p_1^c = 1$ to the long-range value $p_1^c = 0$. The correspondence of $p_1^c = 0$ in the Ising problem is $T_c = \infty$ which is realised only when $0 \leq s \leq 1$. The interesting region $1 < s \leq 2$ of the Ising problem where the transition occurs at a finite temperature is not found in the percolation problem.

In summary, using the transfer-matrix method, we have found the critical behaviour of a one-dimensional bond percolation system with bonds connecting $L$th nearest neighbours. A long-range percolation model is proposed by assuming $p_n = p_1/n^s$ and taking the $L \to \infty$ limit. Our results show that when $s > 2$ only the short-range percolation exists; namely, the system percolates only when $p_1 = 1$. A transition to a long-range percolation is found at $s = 2$ where the percolation threshold drops suddenly from the short-range value $p_1^c = 1$ to the long-range value $p_1^c = 0$. A transition at finite $p_1^c$ is not found in the percolation system.

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## References

Abramowitz M and Stegun I A 1970 *Handbook of Mathematical Functions* (New York: Dover) p 807
Derrida B and Vannimenus J 1980 *J. Physique Lett.* **41** L473–6
Domb C 1949 *Proc. R. Soc.* A **196** 36–51
Dyson F J 1969 *Commun. Math. Phys.* **12** 91–107
Essam J W 1980 *Rep. Prog. Phys.* **43** 833–912
Fröhlich J and Spencer T 1982 *Commun. Math. Phys.* **84** 87–101
Kasteleyn P W and Fortuin C M 1969 *J. Phys. Soc. Japan* (Suppl.) **26** 11–4
Klein W, Stanley H E, Redner S and Reynolds P J 1978 *J. Phys. A: Math. Gen.* **11** L17–22
Li T C, Zhang Z Q and Pu F C 1983 *J. Phys. A: Math. Gen.* **16** 665–70
Pearl M 1973 *Matrix Theory and Finite Math.* (New York: McGraw-Hill) p 285
Reynolds P J, Stanley H E and Klein W 1980 *Phys. Rev.* B **21** 1223–45
Rosenblatt M 1962 *Random Processes* (Oxford: Oxford University Press) p 46
Ruelle D 1968 *Commun. Math. Phys.* **9** 267
Stauffer D 1979 *Phys. Rep.* **54** 1–74
Thouless D J 1969 *Phys. Rev.* **187** 732–3
Zhang Z Q, Pu F C and Li T C 1983 *J. Phys. A: Math. Gen.* **16** 125
Zhang Z Q and Shen J L 1982 *J. Phys. A: Math. Gen.* **15** L363–8